# § Geodesics on Surfaces in $\mathbb{R}^3$

Recall: A curve $\alpha : I \to S \subseteq \mathbb{R}^3$ is a geodesic

iff $\quad \nabla_{\alpha'} \alpha' \equiv 0$

i.e. $\quad (\alpha'')^T = \nabla_{\alpha'} \alpha' \equiv 0$.

In local coordinates, it can be expressed a system of non-linear $2^{nd}$ order ODEs:

$$\boxed{\frac{d^2 u^k}{dt^2} + \Gamma^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} = 0} \quad \text{—— } (*)$$

Fundamental Theorem of geodesics:

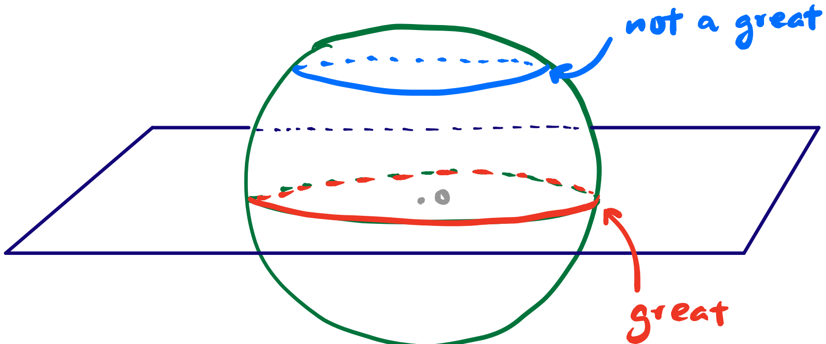
$(*)$ is uniquely solvable (on a short time interval) with any prescribed initial position and initial velocity.

Remark: However, explicit solutions can be very hard to compute analytically. We now look at a few simple examples that allow us to simplify the calculations by making use of the symmetries!

## Example 1: Geodesics on round spheres.

Let $S = \mathbb{S}^2 = \{x^2+y^2+z^2=1\}$ be the unit sphere.

Prop: The geodesics on $\mathbb{S}^2$ are given exactly by (segments) of the "great circles", i.e. circles obtained by intersecting $\mathbb{S}^2$ with a plane passing through the origin.




Proof: Suppose $\alpha(s) : I \to \mathbb{S}^2$ is a geodesic p.b.a.l.

$\alpha$ lies on $\mathbb{S}^2 \Rightarrow \|\alpha\|^2 \equiv 1$ ...... (1)

$\alpha$ p.b.a.l. $\Longrightarrow \|\alpha'\|^2 \equiv 1$ ...... (2)

$\alpha$ geodesic $\Longrightarrow (\alpha'')^T \equiv 0$ ...... (3)

Differentiate (1), $\langle \alpha, \alpha' \rangle \equiv 0$

Differentiate this again and use (2) :

$$\langle \alpha, \alpha'' \rangle + \underbrace{\langle \alpha', \alpha' \rangle}_{\substack{\parallel\!\parallel \\ 1}} \equiv 0$$

$$\Rightarrow \qquad \langle \alpha'', \alpha \rangle \equiv -1$$

Recall that : $T_p S^2 \perp p$

$$\therefore \quad \alpha'' = (\alpha'')^T + (\alpha'')^N$$

$$= \underbrace{(\alpha'')^T}_{\substack{\text{geodesic} \;\; \parallel\!\parallel \\ 0}} + \underbrace{\langle \alpha'', \alpha \rangle}_{\substack{\parallel\!\parallel \\ -1}} \alpha$$

Hence, we arrive at the equation : $\boxed{\alpha'' + \alpha \equiv 0}$ —— (#)

Note: $\boxed{\alpha(s) = p \cos s + q \sin s}^{*}$ solves (#)

for any $p, q \in S^2$ s.t. $p \perp q$ (Exercise : Check this)

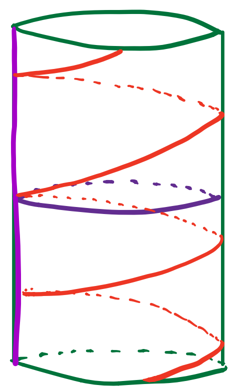
By choosing $p, q$ appropriately, it solves (#) with any given initial position $\alpha(0)$ and initial velocity $\alpha'(0)$ .

Hence, by uniqueness, these are all the possible solutions to the geodesic equation. Finally, notice that $*$ parametrizes a great circle lying in the plane ($\cap S^2$) spanned by $p$ and $q$ .

$\square$

## Example 2: Geodesics on a cylinder

Let $S = \{x^2+y^2=1\}$ be a right circular cylinder (of radius 1)



**Prop:** The geodesics are given by segments of

either
- horizontal circle
- vertical line
- helix

We will give 2 proofs of this.

**Proof 1:** (Make use of symmetry)

As before, let $\alpha : I \to S$ be a geodesic p.b.a.l.

Suppose $\alpha(s) = (x(s), y(s), z(s))$, $s \in I$

$\alpha$ lies on $S$ $\Rightarrow$ $x(s)^2 + y(s)^2 \equiv 1$

$\xRightarrow{\text{differentiate}}$ $xx' + yy' \equiv 0$

$\xRightarrow{\text{differentiate}}$ $xx'' + yy'' + \underbrace{(x')^2 + (y')^2}_{= 1-(z')^2 \quad (\because \text{ p.b.a.l.})} \equiv 0$.

Hence, $\boxed{xx'' + yy'' = (z')^2 - 1}$ —— (##)

**Recall:** $T_{(x,y,z)}S \perp (x,y,0)$

geodesic equation: $(\alpha'')^T \equiv 0 \Rightarrow (x'', y'', z'') \,//\, (x,y,0)$

In other words, $\exists$ function $\lambda(s)$ s.t.

$$\begin{cases} x''(s) = \lambda(s)\, x(s) \\ y''(s) = \lambda(s)\, y(s) \\ z''(s) = 0 \end{cases} \qquad \text{———} \quad (**)$$

Now, we solve $(**)$ with initial conditions:

$$\alpha(0) = (x(0), y(0), z(0)) = (1, 0, 0)$$

$$\alpha'(0) = (x'(0), y'(0), z'(0)) = (0, a, b) \perp \alpha(0)$$

where $a^2 + b^2 = 1$ p.b.a.l.

Solving for $z$, we have $z(s) = bs$ $(\Rightarrow z' \equiv b)$

Put everything back into $(\#\#)$

$$\lambda(s) = x(s)\, x''(s) + y(s)\, y''(s) = b^2 - 1 = -a^2.$$

$(\because x^2 + y^2 \equiv 1)$

Constant!

Solve $x'' = -a^2 x$ with $x(0) = 1$, $x'(0) = 0$

$$x(s) = \cos as$$

Solve $y'' = -a^2 y$ with $y(0) = 0$, $y'(0) = a$

$$y(s) = \sin as$$

In summary, we have

$$\alpha(s) = (\cos as, \sin as, bs) \qquad s \in I$$

where $a, b \in \mathbb{R}$ are constants s.t. $a^2 + b^2 = 1$.

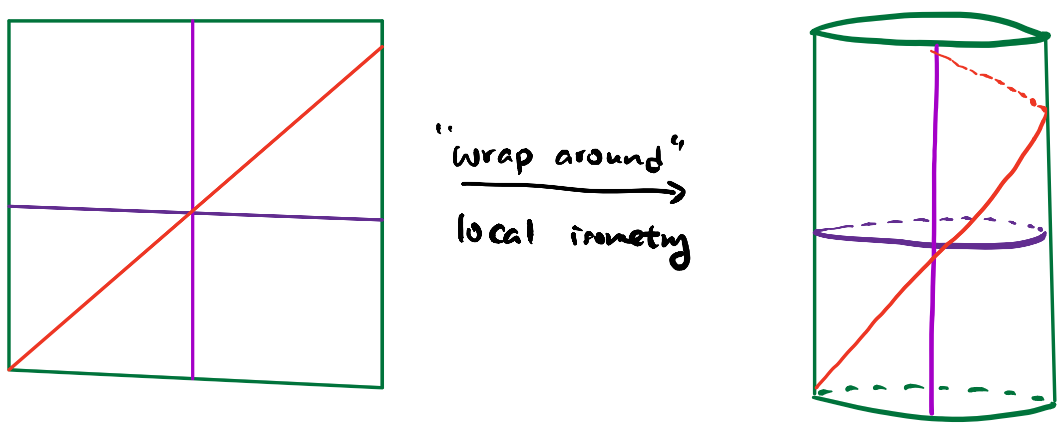
<u>Case 1</u>: $b = 0$, $a = 1$ $\Rightarrow$ horizontal circle

<u>Case 2</u>: $b = 1$, $a = 0$ $\Rightarrow$ vertical line

<u>Case 3</u>: $b \neq 0$, $a \neq 0$ $\Rightarrow$ helix

$\square$

<u>Proof 2</u>: Geodesics are intrinsic concepts, thus is preserved by (local) isometries.




$\square$

## § First Variation Formula for arc length

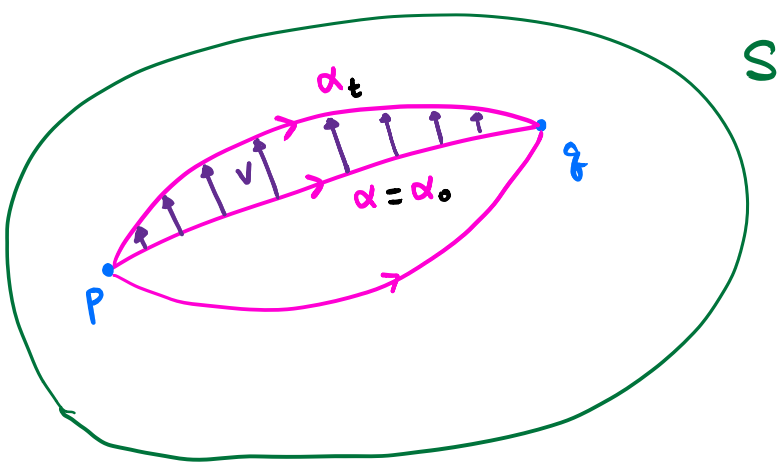
Let $\alpha : [0, L] \to S$ be a curve on $S$ p.b.a.l.

with end points $p = \alpha(0)$, $q = \alpha(L)$.

Consider a "variation" of $\alpha$ with fixed endpoints:

$$\{\alpha_t(s)\}_{t \in (-\varepsilon,\varepsilon)} \quad \text{s.t.} \quad \alpha_0 = \alpha$$

$$\alpha_t : [0, L] \longrightarrow S \qquad \text{and} \quad \begin{aligned} \alpha_t(0) &= \alpha_0(0) \\ \alpha_t(L) &= \alpha_0(L) \end{aligned} \quad \text{for all } t \in (-\varepsilon,\varepsilon)$$




where

$$V(s) = \frac{d}{dt}\Big|_{t=0} \alpha_t(s)$$

variation vector field

(tangent to $S$)

**First Variation Formula:**

$$\boxed{\frac{d}{dt}\Big|_{t=0} \text{Length}(\alpha_t) = -\int_0^L \langle V(s), \alpha''(s) \rangle \, ds}$$

**Remark:** If $\alpha$ is a critical point for $\underline{\underline{\text{ALL}}}$ variations,

then
$$0 = \frac{d}{dt}\Big|_{t=0} \text{Length}(\alpha_t) = -\int_0^L \langle V(s), \alpha''(s) \rangle \, ds$$

for $\underline{\underline{\text{ALL}}}$ variation field $V$ (which is tangential)

$$\Longrightarrow \quad \boxed{(\alpha'')^T \equiv 0}$$

Hence, geodesics are exactly the critical points to the arc length functional!

**Proof of 1st Variation Formula:**

By definition of arc length, (Note: $\alpha_t(s)$ is NOT p.b.a.l. except for $t=0$)

$$\text{Length}(\alpha_t) = \int_0^L \| \alpha_t'(s) \| \, ds$$

Differentiate w.r.t. $t$.

$$\frac{d}{dt} \text{Length}(\alpha_t) = \int_0^L \frac{1}{\| \alpha_t'(s) \|} \left\langle \frac{\partial}{\partial t}\left(\frac{\partial \alpha}{\partial s}\right), \frac{\partial \alpha}{\partial s} \right\rangle ds$$

switch

$$= \int_0^L \frac{1}{\| \alpha_t'(s) \|} \left\langle \frac{\partial}{\partial s}\left(\frac{\partial \alpha}{\partial t}\right), \frac{\partial \alpha}{\partial s} \right\rangle ds$$

Evaluate at $t=0$, $\| \alpha_0'(s) \| \equiv 1$, $\left.\frac{\partial \alpha}{\partial t}\right|_{t=0} = V$

$$\frac{d}{dt}\Big|_{t=0} \text{Length}(\alpha_t) = \int_0^L \left\langle \frac{\partial}{\partial s} V, \frac{\partial}{\partial s}\alpha \right\rangle ds$$

$$= -\int_0^L \langle V, \alpha'' \rangle ds$$

$$+ \cancel{\left\langle V, \frac{\partial}{\partial s}\alpha \right\rangle \Big|_{s=0}^{s=L}} = 0$$

(fixed end points)

$\square$